

Topologically protected helicity cascade in non-Abelian quantum turbulence

Michikazu Kobayashi¹ and Masahito Ueda^{2,3}

¹*Department of Physics, Kyoto University, Oiwake-cho,
Kitashirakawa, Sakyo-ku, Kyoto 606-8502, Japan,*

²*Department of Physics, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan,*

³*RIKEN Center for Emergent Matter Science (CEMS), Wako, Saitama 351-0198, Japan*

(Dated: June 24, 2016)

By numerically studying non-Abelian quantum turbulence, we find that the helicity cascade is topologically protected against reconnection of vortices and leads to the energy spectrum $E(k) \propto k^{-7/3}$ with a large-scale network of non-Abelian vortices. Our prediction can be tested in the cyclic phase of a spin-2 spinor Bose-Einstein condensate.

Introduction.— Turbulence is a highly nonequilibrium and dynamically nonstationary phenomenon that appears in a variety of macroscopic systems [1]. Yet it exhibits stationary statistical laws, culminating in Kolmogorov's $-5/3$ law [2, 3] in an inertial range where the energy flows to smaller scales with a constant flux, which is known as the energy cascade.

Turbulence is characterized by the universality class, the conservation law, and the cascade of a physical quantity in the inertial range. In Kolmogorov's law, the kinetic energy is the conserved quantity. In two dimensions, there is another conserved quantity – the enstrophy which is defined as the square of the fluid vorticity. Two-dimensional steady turbulence therefore belongs to a different universality class supported by the enstrophy cascade, featuring the energy spectrum $E(k) \propto k^{-3}$ at small scales and Kolmogorov's $-5/3$ law at large scales [4]. In three dimensions, there is yet another conserved quantity – the helicity which is given as the inner product of the fluid velocity and vorticity [5, 6]. However, its contribution to turbulence is too small and has never been observed as a statistical law in three-dimensional steady turbulence. This fact has remained a mystery in the studies of turbulence [7, 8].

Quantum turbulence exhibits a spatially and dynamically complex structure of quantized vortices [9]. Like classical turbulence, quantum turbulence also features Kolmogorov's $-5/3$ law associated with the energy cascade [10]; however it is expected to show various universality classes depending on the topological nature of quantized vortices. In this Letter, we propose non-Abelian quantum turbulence as a new theoretical paradigm of turbulence, and report the universality class characterized by the topologically protected helicity cascade. Non-Abelian vortices are defined as those having non-Abelian topological charges classified by the fundamental group [11]. The sharp distinction between Abelian and non-Abelian vortices manifests itself in the collision dynamics. The collision of two Abelian vortices mostly ends up with reconnection [12] and causes a discrete change in the helicity. Non-Abelian vortices, on the contrary, do not reconnect due to the topological constraint [13], and hence the helicity is *topologically* protected. A unique universality class of turbulence is shown to emerge as a consequence of the helicity cascade

in contrast to Abelian quantum turbulence characterized by the energy cascade.

As a simplest superfluid system that supports non-Abelian vortices and for possible experimental implementation, we consider the cyclic phase of a spin-2 Bose-Einstein condensate (BEC) and its turbulent state with energy injection and weak dissipation. We find a unique power-law energy spectrum $E(k) \propto k^{-7/3}$ in the inertial range instead of Kolmogorov's law in Abelian quantum turbulence [10], and show that this power-law spectrum is directly connected to the helicity cascade. We also show that the helicity of a closed non-Abelian vortex loop is conserved without reconnection. These findings unveil the universality class characterized by the $-7/3$ power law due to the topologically protected helicity cascade. We have also performed simulations in other BEC systems such as spin-1 spinor BECs and a two-component BEC with and without the quadratic Zeeman effect and found no evidence of this peculiar power law.

Model.— The Hamiltonian \mathcal{H} of the spin-2 BEC is given by [14]

$$\mathcal{H} = \int d^3x \left\{ \frac{\hbar^2}{2M} \sum_{m=-2}^2 |\nabla \psi_m|^2 + \frac{g_0}{2} (\rho - \bar{\rho})^2 + \frac{g_1}{2} \mathbf{S}^2 + \frac{g_2}{2} |A|^2 \right\}, \quad (1)$$

where $\psi = (\psi_2, \psi_1, \psi_0, \psi_{-1}, \psi_{-2})^T$ is the spinor order parameter in the irreducible representation (T denotes the transpose), M is the atomic mass, $\rho = \sum_{m=-2}^2 |\psi_m|^2$ is the total number density with $\bar{\rho}$ being its mean, and $\mathbf{S} = \sum_{m,n=-2}^2 \psi_m^* [\hat{\mathbf{S}}]_{m,n} \psi_n$ and $A = \sum_{m=-2}^2 \psi_m \psi_{-m}$ are the spin vector density and the spin-singlet pair amplitude, respectively with $\hat{\mathbf{S}}$ being the vector of spin-2 matrices. The ground state for the positive coupling constants $g_{0,1,2} > 0$ belongs to the cyclic phase where both \mathbf{S} and A vanish [15]. A representative state of the cyclic phase is given by $\psi_{\text{cyclic}} = \sqrt{\bar{\rho}} e^{i\phi} e^{-i\hat{\mathbf{S}} \cdot \mathbf{n}\theta} (i/2, 0, 1/\sqrt{2}, 0, i/2)^T$ [16], where $\phi \in [0, 2\pi/3)$ is the $U(1)$ phase, and \mathbf{n} and θ are the unit vector of the rotation axis and the rotation angle, respectively. Quantized vortices appear as topological excitations and play a major role in quantum turbulence. For the cyclic phase, vortices can have

zero and fractional circulations 0, $h/(3M) \equiv \kappa_{\text{cyclic}}$, and $2\kappa_{\text{cyclic}}$, where the former two make major contributions to turbulence because the last one is energetically costly [17].

Simulation of turbulence.— The dynamics of the system is governed by the nonlinear Schrödinger equation (NLSE): $i\hbar\partial_t\psi_m = \delta\mathcal{H}/\delta\psi_m^*$. To obtain a statistical steady state of turbulence, we need to introduce energy injection and dissipation. Dissipation is introduced by replacing $i\hbar$ with $i\hbar - \gamma$ in the left-hand side of NLSE, where γ is a dissipation constant which we choose to be $\gamma = 0.05\hbar$. Energy injection is introduced by a external current \mathbf{v}_{ext} through the non-uniform Galilean transformation: $\partial_t \rightarrow \partial_t - \mathbf{v}_{\text{ext}} \cdot \nabla$. We perform numerical simulations in the periodic box with 1024^3 grids. The space is discretized by the grids with the distance $\Delta x = 0.5\xi$, where the healing length $\xi \equiv \hbar/\sqrt{2Mg_0\rho}$ has the same order of magnitude as a vortex core size. The coupling constants are chosen to be $g_{1,2} = 0.5g_0$ in the present simulations. Smaller values of $g_{1,2}$ [18] would decrease the energy gap between the cyclic phase and other metastable phases. Therefore, we should use a weak energy injection and a large system with dilute vortices to maintain their non-Abelian character in the stable cyclic phase. The external current \mathbf{v}_{ext} is determined as the divergence-free part of a large-scale energy current $v_{i=x,y,z} = \bar{v} \sum_{0 \leq |\mathbf{n}| \leq 2} \cos(2\pi\mathbf{n} \cdot \mathbf{x}/L + \theta_{\mathbf{n},i})$, where $\mathbf{n} \in \mathbb{Z}^3$, $\bar{v} = 0.4\sqrt{g_0\rho}/M$, and the system size $L = 512\xi$. Here $\theta_{\mathbf{n},i}$ take uniformly distributed random values between 0 and 2π for each \mathbf{n} and $i = x, y, z$. We have used the pseudo-spectral method in space and the fourth-order Runge-Kutta method in time.

After a long-time evolution, the statistical steady state of quantum turbulence with a large number of vortices is achieved as shown in Figs. 1 (a)-(d). As a comparison, we also perform numerical simulations for a scalar BEC with every vortex having the unit circulation $\kappa_{\text{scalar}} = h/M$ under the same condition and obtain the corresponding statistical steady state as shown in Fig. 1 (f)-(i). We use $\bar{v} = 0.8\sqrt{g_0\rho}/M$ and obtain almost the same total vortex length as that for the cyclic state. We calculate the kinetic energy spectrum $E(k)$ per unit mass defined as [19]

$$\int dk E(k) = \frac{1}{2L^3} \int d^3x \mathbf{v}^2 \equiv E_v, \quad (2)$$

where \mathbf{v} is the superfluid velocity defined as $\rho\mathbf{v} = \bar{\kappa} \sum_{m=-2}^2 \text{Im}[\psi_m^* \nabla \psi_m]$ with $\bar{\kappa} \equiv \hbar/M$ [14]. The energy spectrum $E(k)$ for the cyclic state of the spinor BEC is plotted as a blue curve in Fig. 2 which shows a power-law for $k < l^{-1}$, where $l = 1/\sqrt{\rho_{\text{vortex}}}$ is the mean intervortex spacing with the vortex length density ρ_{vortex} . Compared with the energy spectra $E(k)$ for the scalar BEC (green curve) which shows Kolmogorov's law for $k < l^{-1}$ [20], $E(k)$ for the cyclic phase clearly deviates from Kolmogorov's law and its slope is steeper. This result suggests that the energy resides in large-scale network structures of vortices. Fitting the data in Fig.

2 with $E(k) \propto k^{-\eta}$ for $8\pi/L < k < 48\pi/L$, we find $\eta \simeq 2.28(5)$ which is consistent with $7/3$.

Helicity cascade.— We argue that the physics behind the $-7/3$ power law spectrum is the helicity cascade. In the inertial range far from the energy-injecting range and the energy-dissipative range, we suppose that helicity H defined as $H \equiv (1/L^3) \int d^3x \mathbf{v} \cdot (\nabla \times \mathbf{v})$ is approximately conserved with neither effect of energy injection nor dissipation, and cascades to the region at higher wavenumbers. Because helicity has the dimension $[LT^{-2}]$, the helicity flux ε_H per unit time in the wavenumber space has the dimension $[LT^{-3}]$. Provided that the energy spectrum has a scale invariant power-law structure and is determined only from the helicity flux ε_H in the inertial range, we obtain the energy spectrum $E(k) \propto \varepsilon_H^{2/3} k^{-7/3}$ which is consistent with the result of our numerical simulation in Fig. 2.

Now the vorticity $\nabla \times \mathbf{v}$ vanishes everywhere except for the vortex core as $\nabla \times \mathbf{v} = \kappa_{\text{cyclic}} \delta(\mathbf{x} - \mathbf{x}_{\text{vortex}}) \mathbf{e}_{\parallel}$ [13], where $\mathbf{x}_{\text{vortex}}$ is the position of the vortex core and \mathbf{e}_{\parallel} is the tangential vector of a vortex line. The helicity is, therefore, obtained as the one-dimensional integral along the vortex line: $H = (\bar{\kappa}\kappa_{\text{cyclic}}/L^3) \int ds \partial_{\parallel} \phi(\mathbf{x} = \mathbf{x}_{\text{vortex}}(s) + \epsilon)$, where s is defined in the parametric form of the position of the vortex core $\mathbf{x}_{\text{vortex}} = \mathbf{x}_{\text{vortex}}(s)$, ∂_{\parallel} is the derivative along \mathbf{e}_{\parallel} at $\mathbf{x}_{\text{vortex}}(s)$, and ϵ is a small vector. When there are only vortex loops in the system, the contribution of each loop to the helicity should be an integer multiple of $\kappa_{\text{cyclic}}^2/L^3$. In order for a vortex loop to have a nonzero helicity, a vortex line should pierce the vortex loop because the $U(1)$ phase ϕ changes by $\pm 2\pi/3$ around the vortex loop. Thus, we can infer that simple vortex rings do not tend to have a finite helicity and knotted (or linked) vortex loops tend to have the helicity.

Quantized vortices are characterized by their topological charges classified by the fundamental group of the order-parameter manifold. The fundamental group of the cyclic phase of the spinor BEC is a non-Abelian tetrahedral group, whereas that of the scalar BEC is an Abelian additive group of integers. The vortex dynamics crucially depends on whether vortices are Abelian or non-Abelian. Reconnection and passing through of non-Abelian vortices are forbidden due to the topological constraint, and vortices maintain their topological structure, leading to the conservation of helicity. On the other hand, Abelian vortices can change their topological structure through reconnection or passing through of vortices. As a consequence, knotted (or linked) loops with a nonzero helicity in the scalar BEC unravel, and the helicity tends to vanish. To demonstrate this, we consider the dynamics starting from a closed trefoil knot of a vortex which has a nonzero helicity $HL^3 = 3\kappa_{\text{cyclic}}^2$. The trefoil knot of a non-Abelian vortex maintains its topological structure as shown in Figs. 3 (a)-(e), because reconnection and passing through are topologically prohibited. As a result, the trefoil knot of a non-Abelian vortex is topologically stable. The blue line in Fig. 3 (k) shows the numerically calculated helicity which stays at a nonzero

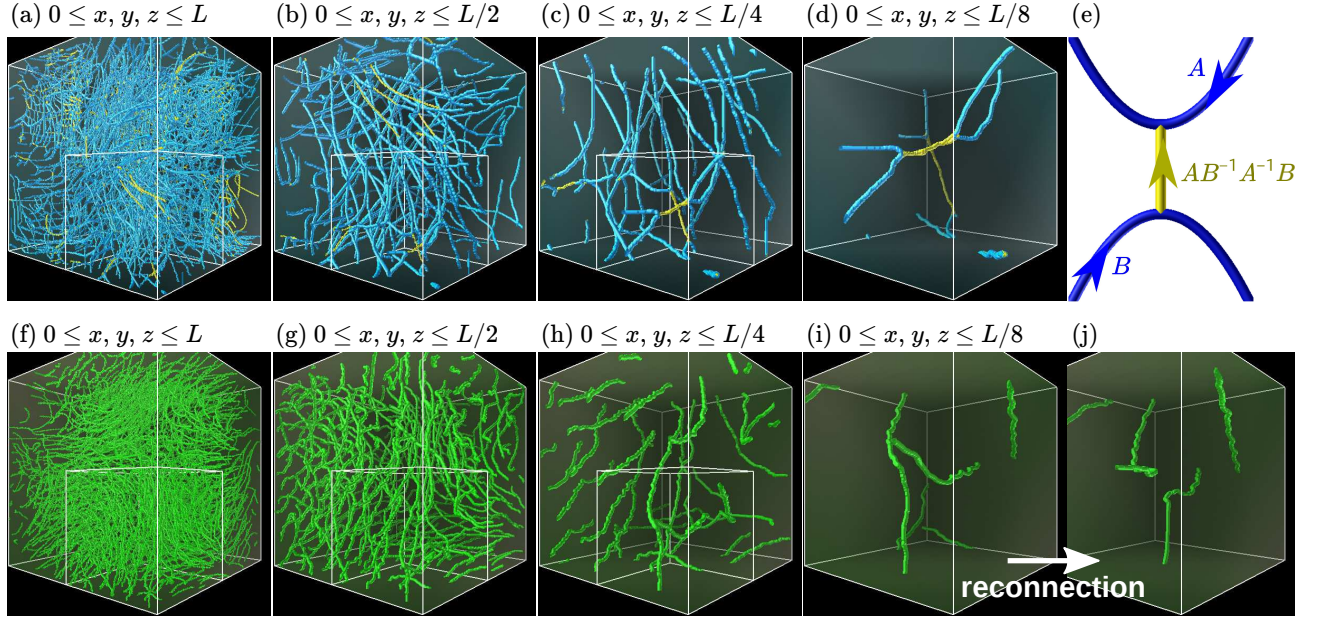


FIG. 1. Snapshots of quantum turbulence of a non-Abelian spin-2 spinor BEC (upper panels) compared with those of an Abelian scalar BEC (lower panels). In upper panels, blue and yellow curves show vortices with circulations $\kappa_{\text{cyclic}} = h/(3M)$ and 0, respectively. In lower panels, green curves show vortices with circulation $\kappa_{\text{scalar}} = h/M$. The leftmost pair of upper and lower panels show vortices in the entire region ($0 \leq x, y, z \leq L$) and each of the following pair of upper and lower panels show the enlarged views: $0 \leq x, y, z \leq L/2$ in (b) and (g), $0 \leq x, y, z \leq L/4$ in (c) and (h), and $0 \leq x, y, z \leq L/8$ in (d) and (i). In upper panels, blue-colored vortices are bridged by yellow-colored vortices, forming a large-scale network of vortices. A rung formation is clearly seen in (d) as a consequence of the non-commutative nature of the two colliding vortices as schematically illustrated in (e), where two non-Abelian vortices with the circulation κ_{cyclic} having topological charges A and B are bridged with a rung vortex whose topological charge is $AB^{-1}A^{-1}B$. Note that this charge would become trivial (i.e. equal to 1) for Abelian vortices for which A and B commute. In lower panels, green colored vortices are not connected unlike non-Abelian vortices because they can reconnect when they collide. In (i), two parallel vortices reconnect and change their spatial configuration as shown in (j).

value of $3\kappa_{\text{cyclic}}^2/L^3$. For the dynamics of a trefoil knot of an Abelian vortex [21] shown in Figs. 3 (f)-(j), on the other hand, reconnection occurs at $t \simeq 50$ (Fig. 3 (g)), but the trefoil maintains its structure (Fig. 3 (h)). Reconnection again occurs at $t \simeq 140$ (Fig. 3 (i)), and the trefoil disintegrates into two disconnected rings (Fig. 3 (j)). We see that a change in the topological configuration of a vortex loop is directly connected to a change in the helicity as shown in Fig. 3 (k) with a green line. The helicity remains unchanged for the first reconnection at $t \simeq 50$ and suddenly vanishes upon the second reconnection at $t \simeq 140$. Thus, when an external energy injection generates large vortex loops with nonzero helicities, they keep their topological structures invariant and cascade to smaller scales in non-Abelian quantum turbulence. In Abelian quantum turbulence, on the other hand, vortex loops lose helicity by breaking up into several smaller vortex rings.

Concluding remarks.— We have numerically studied the steady state of fully developed non-Abelian quantum turbulence. The obtained kinetic energy spectrum shows a unique $-7/3$ power law. This exponent is directly related to the conservation law of the helicity in the inertial range and is absent or less important in classical turbu-

lence. To support the validity of the helicity cascade in non-Abelian quantum turbulence, we have also checked that a knotted non-Abelian vortex does not reconnect, maintaining its topological structure and conserving the helicity unlike an Abelian vortex. This is the first observation of the universality class originating from the helicity cascade.

Two important questions have remained unresolved: (i) why is the helicity cascade more important than the energy cascade in non-Abelian quantum turbulence despite the fact that both helicity and energy are conserved, and (ii) what is the relationship between the helicity and the topological structure of vortex lines? The key to the first question is the fact that energy arises from both mass and spin currents, whereas helicity arises only from mass current. We expect that the second question can be resolved by examining the close relationship between the helicity and the linking number of vortex lines [22]. Figures 3 (a)-(e) indeed show that the change from the trefoil knot (linking number is 3) to two disconnected loops (linking number is 0) can be related to the change of the helicity from $3\kappa_{\text{scalar}}^2/L^3$ to 0. If this conjecture is true, we may interpret the helicity cascade as the linking-number cascade in non-Abelian quantum turbulence.

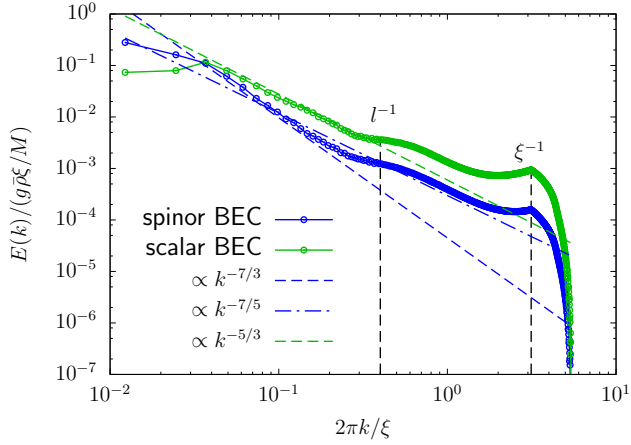


FIG. 2. Energy spectra $E(k)$ of the fluid velocity for turbulence of a spin-2 BEC (blue curve) and a scalar BEC (green curve). The green dashed line shows Kolmogorov's law $\propto k^{-5/3}$ due to the energy cascade for $k < l^{-1}$. The blue dashed line shows the spectrum $E(k) \propto k^{-7/3}$ for the helicity cascade at $k < l^{-1}$ (see the main text). The blue dash-dotted line shows the spectrum $E(k) \propto k^{-7/5}$ for $l^{-1} < k < \xi^{-1}$. Here $l = \sqrt{1/\rho_{\text{vortex}}}$ is the mean inter-vortex spacing and takes $l \simeq 15.7\xi$ for the spinor BEC and $l \simeq 15.5\xi$ for the scalar BEC, and $\xi = \hbar/\sqrt{2Mg_0\rho}$ is the healing length. The ensemble average is taken over 500 steady states generated for different \mathbf{v}_{ext} .

We also point out that there appears another power-law structure in the region of $l^{-1} < k < \xi^{-1}$ for both Abelian and non-Abelian cases as shown in Fig. 2 with a blue dash-dotted line. Exponents in Abelian and non-Abelian cases are estimated to be 1.42(5) and 1.49(20) respectively, both of which are very close to 7/5. In the region of $l^{-1} < k < \xi^{-1}$, the single-vortex dynamics such as the Kelvin-wave cascade plays an important role, and the obtained exponent 7/5 is consistent with one of the theoretically predicted exponents [10].

Our observation may help us understand why the helicity cascade is less important in classical fluid turbulence, where vortex reconnections in the NLSE occur as singular dynamics of vortices and similar singular behavior occurs in classical fluid turbulence as a weak solution of the Euler equation [23] in which higher derivatives of the velocity field cannot be defined. Furthermore, there are other systems in which non-Abelian topological defects appear, such as crystals (dislocation and disclination) [24], liquid crystals (disclination) [25], and cosmology (cosmic strings) [26, 27], and our result may have implications for their non-equilibrium dynamics.

ACKNOWLEDGMENTS

We would like to thank the organizers of the “Core-to-Core International meeting χ Mag 2016 Symposium”

conference held February 21-24, 2016 at Hiroshima,

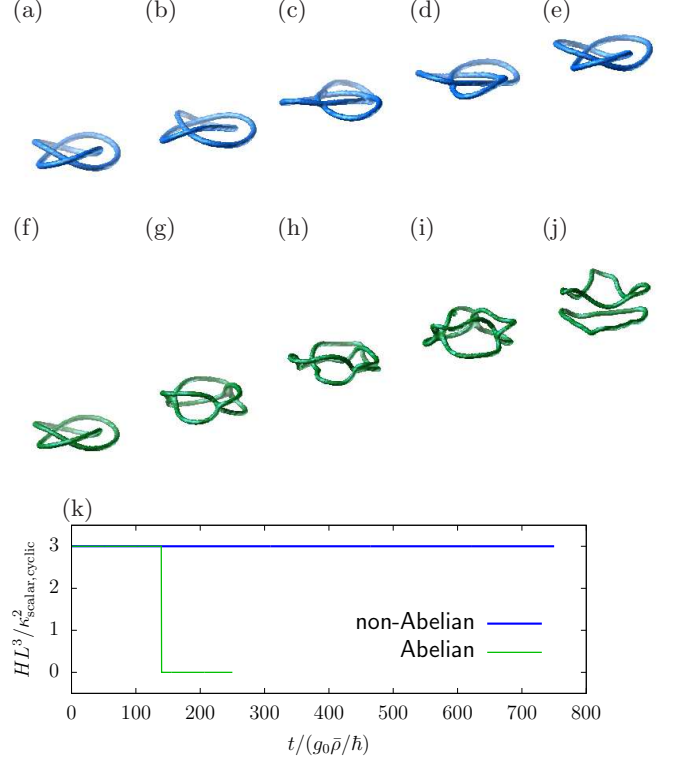


FIG. 3. (a)-(e): Dynamics of a non-Abelian trefoil knot at (a) $t = 0$, (b) $t = 150$, (c) $t = 300$, (d) $t = 420$, and (e) $t = 540$ in units of $g_0 \bar{\rho} / \hbar$. (f)-(j): Dynamics of an Abelian trefoil knot at (f) $t = 0$, (g) $t = 50$, (h) $t = 100$, (i) $t = 140$, and (j) $t = 180$ in units of $g_0 \bar{\rho} / \hbar$. Since a vortex knot has the self-induced velocity proportional to the mean radius, it moves upward in time. The Abelian knot disintegrates into two vortex rings upon the second reconnection as shown in (j). (k): Time dependence of helicity for a non-Abelian trefoil knot (blue) and that of an Abelian trefoil knot (green).

where the present study was initiated, and Shin-ichi Sasa for the helpful suggestions and comments. The work of MK is supported in part by Grant-in-Aid for Scientific Research No. 26870295 and by Grant-in-Aid for Scientific Research on Innovative Areas “Fluctuation & Structure” (No. 26103519) from the Ministry of Education, Culture, Sports, Science and Technology of Japan. MU acknowledges the support by KAKENHI Grant No. 26287088 from the Japan Society for the Promotion of Science and a Grant-in-Aid for Scientific Research on Innovative Areas “Topological Materials Science” (KAKENHI Grant No. 15H05855) and the Photon Frontier Network Program from MEXT.

-
- [1] U. Frisch, “*Turbulence*” (Cambridge University Press, Cambridge, 1995).
- [2] A. N. Kolmogorov, Proc. R. Soc., Sect. A **434**, 9 (1991); A. N. Kolmogorov, Proc. R. Soc., Sect. A **434**, 15 (1991).
- [3] G. K. Batchelor, “*The Theory of Homogeneous Turbulence*” (Cambridge University Press, Cambridge, 1953).
- [4] R. H. Kraichnan, Phys. Fluid, **10**, 1417 (1967).
- [5] W. Thomson, Trans. Roy. Soc. Edin. **25**, 217 (1868).
- [6] H. K. Moffatt, J. Fluid Mech. **35** 117 (1969); H. K. Moffatt and R. L. Ricca, Proc. R. Soc. Lond. A **439**, 411 (1992).
- [7] R. H. Kraichnan, J. Fluid Mech. **59**, 745 (1973).
- [8] V. Borue and S. A. Orszag, Phys. Rev. E **55**, 7005 (1997); Q. Chen, S. Chen, and G. L. Eyink, Phys. Fluids **15**, 361 (2003); Q. Chen, S. Chen, G. L. Eyink, and D. D. Holm, Phys. Rev. Lett. **90**, 214503 (2003); D. O. Gómez and P. D. Mininni, Physica A **342**, 69 (2004); P. D. Mininni, A. Alexakis, and A. Pouquet, Phys. Rev. E **74**, 016303 (2006).
- [9] R. J. Donnelly, “it Quantized Vortices in Helium II” (Cambridge University Press, Cambridge, 1991).
- [10] M. Tsubota, K. Kasamatsu, and M. Kobayashi in “*Novel Superfluid: Quantized vortices in superfluid helium and atomic Bose-Einstein condensates*”, International Series of Monographs on Physics, edited by K. H. Bennemann and J. B. Ketterson (Oxford Science Publications, Oxford, 2013).
- [11] The topological charges of quantized vortices are defined by the fundamental group π_1 of the order-parameter manifold G/H of the system, where G is the symmetry of the Hamiltonian and H is the remaining symmetry of the order parameter. In general, three distinct definitions can be adopted for non-Abelian (Abelian) vortices: (i) G is non-Abelian (Abelian), (ii) H is non-Abelian (Abelian), and (iii) $\pi_1(G/H)$ is non-Abelian (Abelian). We adopt definition (iii) in this Letter because it is directly connected to the collision dynamics of quantized vortices.
- [12] J. Koplik and H. Levine, Phys. Rev. Lett. **71**, 1375 (1993); **76**, 4745 (1996); M. Leadbeater, T. Winiecki, D. C. Samuels, C. F. Barenghi, and C. S. Adams, Phys. Rev. Lett. **86**, 1410 (2001).
- [13] M. Kobayashi, Y. Kawaguchi, M. Nitta, and M. Ueda, Phys. Rev. Lett. **103**, 115301 (2009).
- [14] Y. Kawaguchi and M. Ueda, Phys. Rep. **520** 253 (2010).
- [15] H. Mäkelä, Y. Zhang, and K.-A. Suominen, J. Phys. A **36**, 8555 (2003); H. Mäkelä, J. Phys. A **39**, 7423 (2006); F. Zhou and G. W. Semenoff, Phys. Rev. Lett. **97**, 180411 (2006).
- [16] M. Koashi and M. Ueda, Phys. Rev. Lett. **84**, 1066 (2000); M. Ueda and M. Koashi, Phys. Rev. A **65**, 063602 (2002); H. Saito and M. Ueda, Phys. Rev. A **72**, 053628 (2005).
- [17] In the cyclic phase, ψ_{cyclic} is invariant under the following 12 transformations [15]: $\mathbf{1}$, $I_{x,y,z} = e^{-i\hat{S}_{x,y,z}\pi}$, $C = e^{2\pi i/3} e^{-2\pi i(\hat{S}_x + \hat{S}_y + \hat{S}_z)/(3\sqrt{3})}$, C^2 , $I_{x,y,z}C$, and $I_{x,y,z}C^2$. These 12 transformations form the non-Abelian tetrahedral group. Topological charges of vortices can be classified by 12 elements $\mathbf{1}$, $I_{x,y,z}$, \dots of the non-Abelian tetrahedral group. Details are reviewed in M. Kobayashi, J. Phys. Conf. Ser. **297**, 021013 (2011) for order parameters and circulations of each vortices, and S. Kobayashi, *et. al.*, Nucl. Phys. B **856**, 577 (2012) for the classification of topological charges by the fundamental group.
- [18] A. Widera, F. Gerbier, S. Fölling, T. Gericke, O. Mandel, and I. Bloch, New J. Phys. **8**, 152 (2006).
- [19] E_v in Eq. (2) usually diverges because the velocity \mathbf{v} diverges near the vortex core $\mathbf{x}_{\text{vortex}}$ as $|\mathbf{v}| \propto 1/|\mathbf{x} - \mathbf{x}_{\text{vortex}}|^2$. We therefore define E_v by excluding the vortex core from the integration region in Eq. (2) as
- $$E_v = \frac{1}{2L^3} \int_{|\mathbf{x} - \mathbf{x}_{\text{vortex}}| > \epsilon} d^3x \mathbf{v}^2, \quad (3)$$
- where the cutoff ϵ is taken to be the healing length ξ . The energy spectrum $E(k)$ in the inertial range is independent of the choice of ϵ when it is the order of ξ , though E_v itself depends on it.
- [20] M. Kobayashi and M. Tsubota, Phys. Rev. Lett. **94**, 065302 (2005); J. Phys. Soc. Jpn. **74**, 3248 (2005).
- [21] C. F. Barenghi, Milan J. Math. **75**, 177 (2007).
- [22] In the study of classical fluid, the relationship between the helicity and the linking of the vorticity lines has been discussed by X. Liu and R. L. Ricca, J. Phys. A: Math. Theor. **45** 205501 (2012), and X. Liu and R. L. Ricca, J. Fluid Mech. **773** 34 (2015).
- [23] C. De Lellis and L. Székelyhidi Jr., J. Eur. Math. Soc. **16**, 1467 (2014).
- [24] P. M. Chaikin and T. C. Lubensky, “*Principles of Condensed Matter Physics*” (Cambridge University Press, Cambridge 2000).
- [25] V. Parnaru and G. Toulouse, J. Phys. (Paris) **38** 887, (1977); N. D. Mermin, Rev. Mod. Phys. **51**, 591 (1979).
- [26] A. Vilenkin and E. P. S. Shellard, “*Cosmic Strings and Other Topological Defects*” (Cambridge University Press, Cambridge, 1994); M. B. Hindmarsh and T. W. B. Kibble, Rep. Prog. Phys. **58**, 477 (1995).
- [27] D. N. Spergel and U. L. Pen, Astrophys. J. **491**, L67 (1997); P. McGraw, Phys. Rev. D **57**, 3317 (1998); M. Bucher and D. N. Spergel, Phys. Rev. D **60**, 043505 (1999).